The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geq 7$ *

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Abstract

A minimal surface of general type with $p_g(S) = 0$ satisfies $1 \leq K^2 \leq 9$ and it is known that the image of the bicanonical map φ is a surface for $K_S^2 \geq 2$, whilst for $K_S^2 \geq 5$, the bicanonical map is always a morphism. In this paper it is shown that φ is birational if $K_S^2 = 9$ and that the degree of φ is at most 2 if $K_S^2 = 7$ or $K_S^2 = 8$.

By presenting two examples of surfaces S with $K_S^2 = 7$ and 8 and bicanonical map of degree 2, it is also shown that this result is sharp. The example with $K_S^2 = 8$ is, to our knowledge, a new example of a surface of general type with $p_g = 0$.

The degree of φ is also calculated for two other known surfaces of general type with $p_g = 0$, $K_S^2 = 8$. In both cases the bicanonical map turns out to be birational.

1 Introduction

Many examples of complex surfaces of general type with $p_g = q = 0$ are known, but a detailed classification is still lacking, despite much progress in the theory of algebraic surfaces. Surfaces of general type are often studied using properties of their canonical curves. If a surface has $p_g = 0$, then there are of course no such curves, and it seems natural to look instead at the bicanonical system, which is not empty.

Minimal surfaces S of general type with $p_g(S) = 0$ satisfy $1 \leq K_S^2 \leq 9$. By a result of Xiao Gang [15], the image of the bicanonical map is a surface

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for $K_S^2 \geq 2$, while for $K_S^2 \geq 5$, by Reider's theorem [14], the bicanonical map is always a morphism. Xiao Gang [16] showed that the degree of the bicanonical map is ≤ 2 for surfaces of general type, with a limited number of possible exceptions. Surfaces with $p_g = 0$ are among the exceptional cases, and [16] gives practically no information on the possible degrees of their bicanonical maps.

The first author [9] showed that if $K_S^2 \geq 3$ and the bicanonical map is a morphism, its degree is ≤ 4. There are examples due to Burniat [3] (see also [13]) with $3 \leq K^2 \leq 6$ having bicanonical map of degree 4; indeed, [10] gives a precise description of the surfaces with $K_S^2 = 6$, $p_g(S) = 0$ and bicanonical map of degree 4. Here we refine the result of [9] by proving the following results:

Theorem 1.1 *Let S be a minimal surface of general type defined over $\mathbb{C}$ with $p_g(S) = 0$, and $\varphi\colon S \to \Sigma \subset \mathbb{P}^{K_S^2}$ its bicanonical map, with image Σ.*

(i) If $K_S^2 = 9$ then φ is birational;

(ii) if $K_S^2 = 7, 8$ then φ has degree ≤ 2.

Proposition 1.2 *There exist minimal surfaces of general type with $p_g(S) = 0$, $K_S^2 = 7, 8$ and bicanonical map of degree 2.*

We prove this by giving two examples of surfaces S with $K_S^2 = 7$ and 8 and bicanonical map of degree 2. The example with $K_S^2 = 7$ is due to Inoue [8, Remark 6], who constructed it as a quotient of a complete intersection in the product of four elliptic curves by a free action of $\mathbb{Z}_2^5$. Here we give an alternative description as a $\mathbb{Z}_2^2$-cover of a singular rational surface that allows us to describe the bicanonical map and compute its degree. The example with $K_S^2 = 8$ is obtained by applying a construction of Beauville ([1, p. 123, Exercise 4], and cf. [7]). To the best of our knowledge, it is a new example of a surface of general type with $p_g = 0$.

Only a few examples of surfaces with $p_g = 0$, $K_S^2 = 8$ are known. It is a nontrivial exercise to compute the degree of the bicanonical map for an explicit surface, and for interest, we include the computation for two other known surfaces of general type with $p_g = 0$, $K_S^2 = 8$, for both of which the bicanonical map turns out to be birational.

The paper is organized as follows: Section 2 recalls some facts on irregular double covers, one of the main ingredients of the proof of the theorem; in

Section 3 we prove the main result. For $K_S^2 = 7, 8$ the proof consists of using the methods of Section 2 to exclude the possibility that the bicanonical map has degree 4; for $K_S^2 = 9$ the result is obtained by combining Reider's theorem with an analysis of the Picard group of S. In the final Section 4 we present the two examples to prove Proposition 1.2, and we also compute the degree of the bicanonical map of two other surfaces with $p_g = 0$ and $K_S^2 = 8$.

Notations and conventions We work over $\mathbb{C}$; all varieties are assumed to be compact and algebraic. We do not distinguish between line bundles and divisors on a smooth variety, and use additive and multiplicative notation interchangeably. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim$. The remaining notation is standard in algebraic geometry.

2 Irregular double covers and fibrations

We describe here the key facts used in some proofs in this paper.

Let S be a smooth complex surface, $D \subset S$ a curve (possibly empty) with at worst ordinary double points, and M a line bundle on S with $2M \equiv D$. It is well known that there exists a normal surface Y and a finite degree 2 map $\pi\colon Y \to S$ branched over D such that $\pi_*\mathcal{O}_Y = \mathcal{O}_S \oplus M^{-1}$. The singularities of Y are A_1 points and occur precisely above the singular points of D; thus it makes sense to speak of the canonical divisor, the geometric genus, the irregularity and the Albanese map of Y. We refer to Y as the *double cover defined by the relation* $2M \equiv D$. The invariants of Y are:

$$\begin{aligned} K_Y^2 &= 2(K_S + M)^2, \\ \chi(\mathcal{O}_Y) &= 2\chi(\mathcal{O}_S) + \frac{1}{2}M(K_S + M), \\ p_g(Y) &= p_g(S) + h^0(S, K_S + M). \end{aligned} \tag{2.1}$$

If $p_g(S) = q(S) = 0$, the existence of a double cover $\pi\colon Y \to S$ with $q(Y) > 0$ forces the existence of a fibration $f\colon S \to \mathbb{P}^1$ such that π^{-1} of the general fibre of f is disconnected. More precisely we have:

Proposition 2.1 (De Franchis) *Let S be a smooth surface with $p_g(S) = q(S) = 0$ and $\pi\colon Y \to S$ a double cover with at most A_1 points; if $q(Y) > 0$, then*

(i) the Albanese image of Y is a curve B;

(ii) let $\alpha\colon Y \to B$ be the Albanese fibration. Then there exists a fibration $g\colon S \to \mathbb{P}^1$ and a degree 2 map $p\colon B \to \mathbb{P}^1$ such that $p \circ \alpha = g \circ \pi$.

The possibility of existence of such a double cover often leads to a contradiction, using the following result:

Corollary 2.2 *Let S be a minimal surface of general type with $p_g(S) = q(S) = 0$ and $K_S^2 \geq 3$, and $\pi\colon Y \to S$ a double cover with at most A_1 points. Then $K_Y^2 \geq 16(q(Y) - 1)$.*

Proposition 2.1 is an old result of De Franchis [6], explained and generalized in several ways by Catanese and Ciliberto [5]. Proposition 2.1 and Corollary 2.2 are both stated and proved in [10] for smooth Y, but the proof extends verbatim to the case of A_1 points.

3 Proof of Theorem 1.1

Under the assumptions of Theorem 1.1, the image of the bicanonical map is a surface by [15], and the bicanonical map is a morphism by Reider's theorem [14]. Moreover, since $4K_S^2 = \deg\varphi \deg\Sigma$ and Σ is a nondegenerate surface in $\mathbb{P}^{K_S^2}$, the possible values of $\deg\varphi$ are $1, 2, 4$ for $K_S^2 = 7, 8$ and $1, 2, 3, 4$ for $K_S^2 = 9$.

We prove the theorem by analysing separately the cases $K_S^2 = 7, 8, 9$. In each case we argue by contradiction.

3.1 The case $K_S^2 = 7$

By the above remark, it is enough to show that $\deg\varphi = 4$ does not occur. Assume that φ has degree 4. The bicanonical image Σ is a linearly normal surface of degree 7 in $\mathbb{P}^7$ and its nonsingular model has $p_g = q = 0$. By [11, Theorem 8], Σ is the image of the blowup $\widehat{\mathbb{P}}$ of $\mathbb{P}^2$ at two points P_1, P_2 under its anticanonical map $f\colon \widehat{\mathbb{P}} \hookrightarrow \mathbb{P}^7$. If $P_1 \neq P_2$, then f is an embedding, while if P_2 is infinitely near to P_1 (say) then Σ has an A_1 singularity. In either case, the hyperplane section of Σ can be written as $H \equiv 2l + l_0$, where l is the image on Σ of a general line of $\mathbb{P}^2$ and l_0 is the image on Σ of the strict transform of the line through P_1 and P_2. Notice that l_0 is contained

in the smooth part of Σ. Thus we have $2K_S \equiv 2L + L_0$, where $L = \varphi^* l$ and $L_0 = \varphi^* l_0$.

Lemma 3.1 *L_0 satisfies one of the following possibilities:*

(i) there exists an effective divisor D on S such that $L_0 = 2D$; or

(ii) L_0 is a smooth rational curve with $L_0^2 = -4$; or

(iii) there exist smooth rational curves A and B with $A^2 = B^2 = -3$, $AB = 1$, and $L_0 = A + B$.

Proof Remark first that $K_S L_0 = 2$, $L_0^2 = -4$, and $L_0 = 2(K_S - L)$ is divisible by 2 in $\operatorname{Pic} S$. Let θ be a -2-curve of S; then θ is contracted by φ and thus $L\theta = L_0\theta = 0$. Since L and L_0 are independent elements of the 3-dimensional space $H^{1,1}(S)$, S contains at most one -2-curve. We write $L_0 = C + a\theta$, where C is the strict transform of L_0, θ is a -2-curve and $a \geq 0$ (we set $a = 0$ if S has no -2-curve). The equalities $\theta L_0 = 0$ and $L_0^2 = -4$ imply

$$\theta C = 2a, \quad \text{and} \quad C^2 = -4 - 2a^2. \tag{3.1}$$

If C is irreducible, then $K_S C = 2$ implies $C^2 \geq -4$ and thus $a = 0$ and case (ii) holds. If C is reducible, then $C = A + B$, with A and B irreducible curves such that $K_S A = K_S B = 1$. If $A = B$, then $AL_0 = 2A^2 + a\theta A = 2A^2 + a^2$ is even, because L_0 is divisible by 2, and thus a is even and we are in case (i). If $A \neq B$, then $AB \geq 0$ and $A^2, B^2 \geq -3$; by parity considerations and (3.1) we get $A^2 = B^2 = -3$ and either $AB = 1$, $a = 0$ or $AB = 0$, $a = 1$. The first case corresponds to (iii), while the second does not occur. In fact the intersection matrix of A, B, θ would be negative definite, contradicting the index theorem, since $h^{1,1}(S) = 3$. $\diamond$

In cases (ii) or (iii) of Lemma 3.1, let $\pi\colon Y \to S$ be the double cover given by $2(K_S - L) \equiv L_0$; then the formulas (2.1) give $\chi(Y) = 2$ and $K_Y^2 = 16$. Since the bicanonical map φ maps L onto a twisted cubic, $h^0(S, \mathcal{O}_S(2K_S - L)) = 4$ and thus $p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - L)) = 4$; we thus obtain $q(Y) = 3$, contradicting Corollary 2.2.

In case (i) of Lemma 3.1, consider the étale double cover $\pi\colon Y \to S$ given by $2(K_S - L - D) \equiv 0$; arguing as above, we get that the invariants of Y are

$$K_Y^2 = 14, \quad \chi(\mathcal{O}_Y) = 2, \quad p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - L - D)) = 3,$$

so that $q(Y) = 2$ and we again obtain a contradiction to Corollary 2.2.

Hence $\deg \varphi \neq 4$ and we have proved Theorem 1.1 in case $K_S^2 = 7$.

3.2 The case $K_S^2 = 8$

As in case $K_S^2 = 7$, it is enough to show that $\deg \varphi = 4$ does not occur. If φ has degree 4, then the bicanonical image Σ is a linearly normal surface of degree 8 in $\mathbb{P}^8$ whose nonsingular model has $p_g = q = 0$. By [11, Theorem 8], Σ is either the Veronese embedding in $\mathbb{P}^8$ of a quadric $Q \subset \mathbb{P}^3$ or the image of the blowup $\widehat{\mathbb{P}}$ of $\mathbb{P}^2$ at a point P under its anticanonical map $f : \widehat{\mathbb{P}} \hookrightarrow \mathbb{P}^8$.

In the first case $2K_S \equiv 2A$, where A is the hyperplane section of Q. Then $\eta = K_S - A$ is a nontrivial 2-torsion element in $\operatorname{Pic} S$, since $p_g(S) = 0$. The étale double cover $\pi : Y \to S$ given by $2\eta \equiv 0$ has invariants $\chi(Y) = 2$, $K_Y^2 = 16$. Moreover, $p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(A)) = 4$, so that $q(Y) = 3$. Since $K_Y^2 = 16$, this contradicts Corollary 2.2, and therefore Σ is not the Veronese embedding of a quadric.

If the bicanonical image Σ is the image of $\widehat{\mathbb{P}}$ via the map induced by $|-K_{\widehat{\mathbb{P}}}|$, then the hyperplane section of Σ can be written as $H \equiv 2l + l_0$, where l is the image on Σ of a general line of $\mathbb{P}^2$ and l_0 is the image on Σ of the strict transform of a general line through P. Thus $2K_S \equiv 2L + L_0$, where $L = \varphi^* l$ and $L_0 = \varphi^* l_0$, and $L_0 = \varphi^* l_0$ is smooth by Bertini's theorem. Consider now the double cover $\pi : Y \to S$ given by $2(K_S - L) \equiv L_0$; the formulas (2.1) give $\chi(Y) = 3$ and $K_Y^2 = 24$. Since $p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - L)) = 0 + h^0(S, \mathcal{O}_S(L + L_0)) = 5$, we get $q(Y) = 3$, contradicting Corollary 2.2. Thus Σ is also not the image of $\widehat{\mathbb{P}}$.

Hence $\deg \varphi \neq 4$ and the proof of Theorem 1.1, (ii) is complete.

3.3 The case $K_S^2 = 9$

If $K_S^2 = 9$, then by Poincaré duality, $H^2(S, \mathbb{Z})$ is generated up to torsion by the class of a line bundle L with $L^2 = 1$; thus every divisor on S is numerically a multiple of L, and in particular $K_S \sim 3L$.

Assume by contradiction that φ is not birational; then by Reider's theorem (cf. [2, Theorem 2.1]), for every pair of points $x_1, x_2 \in S$ with $\varphi(x_1) = \varphi(x_2)$ there exists an effective divisor C containing x_1, x_2 such that $K_S C - 2 \leq C^2 < \frac{1}{2} K_S C < 2$. Since $K_S \sim 3L$, the only possibility is that $C \sim L$. We can assume that, as x_1 and x_2 vary, the divisor C varies in an irreducible system of curves, which is linear by the regularity of S. Every curve of $|C|$

is irreducible, since the class of C generates $H^2(S,\mathbb{Z})$ up to torsion, and the general curve of $|C|$ is smooth by Bertini's theorem, since $C^2=1$. Therefore $|C|$ is a linear pencil of curves of genus 3 with one base point. For a general $C\in|C|$ we consider the exact sequence:

$$0\to\mathcal{O}_S(2K_S-C)\to\mathcal{O}_S(2K_S)\to\mathcal{O}_C(2K_S)\to 0. \tag{3.2}$$

Since $2K_S-C\sim K_S+2L$, Kodaira vanishing gives $H^1(S,\mathcal{O}_S(2K_S-C))=0$, and the map $H^0(S,\mathcal{O}_S(2K_S))\to H^0(C,\mathcal{O}_C(2K_S))$ induced by the sequence (3.2) is surjective. So the map $f\colon C\to\mathbb{P}^3$ given by $|\mathcal{O}_C(2K_S)|$ is not birational; it follows that f maps C two-to-one onto a twisted cubic, and thus C is hyperelliptic. If we denote by Δ the g^1_2 of C, then $2K_S|_C\equiv 3\Delta$ and also, by the adjunction formula, $K_S+C|_C\equiv 2\Delta$. So $\eta\equiv 4K_S-(3K_S+3C)\equiv K_S-3C$ is trivial when restricted to C. Moreover $\eta\sim 0$ and so η is a torsion element of $\operatorname{Pic} S$. Since $p_g(S)=0$, η is nonzero. Consider the connected étale cover $\pi\colon Y\to S$ associated to η. Because $\eta|_C=0$, the cover $\pi|_{\pi^{-1}(C)}\colon\pi^{-1}(C)\to C$ is trivial and thus $\pi^{-1}(C)$ is a smooth disconnected curve with each component of self-intersection 1. This contradicts the Index theorem and we have thus proved Theorem 1.1, (i). $\diamond$

4 Examples

This section calculates the degree of the bicanonical map in 4 interesting examples, as discussed in the introduction.

Example 4.1 Starting from the quadrilateral $P_1P_2P_3P_4$ in $\mathbb{P}^2$ of Figure 1, let P_5 be the intersection point of the lines P_1P_2 and P_3P_4 and P_6 the intersection point of P_1P_4 and P_2P_3. Write $\Sigma\to\mathbb{P}^2$ for the blowup of $P_1,\dots,P_6$, and e_i for the exceptional curves of Σ over P_i. Denote by l the pullback of a line.

Write $S_1,\dots,S_4$ for the strict transforms on Σ of the sides P_iP_{i+1} of the quadrilateral $P_1P_2P_3P_4$ (we take subscripts modulo 4); these are the only -2-curves of Σ. The morphism $f\colon\Sigma\to\mathbb{P}^3$ given by $|-K_\Sigma|$ has image a cubic surface $V\subset\mathbb{P}^3$, and f is an isomorphism on $\Sigma\setminus\bigcup S_i$, and contracts each S_i to an A_1 point.

If $A\subset\{P_1,\dots,P_6\}$ consists of 4 points no three of which are collinear, then the linear system of conics through the points of A gives rise to a free pencil on Σ; we denote by f_1 the strict transform of a general conic through

$P_2P_4P_5P_6$, by f_2 that of a general conic through $P_1P_3P_5P_6$ and by f_3 that of a general conic through $P_1P_2P_3P_4$.

Finally, we introduce the "diagonals" of the quadrilateral $P_1P_2P_3P_4$, writing $\Delta_1, \Delta_2, \Delta_3$ for the strict transform of P_1P_3, P_2P_4 and P_5P_6. The divisors

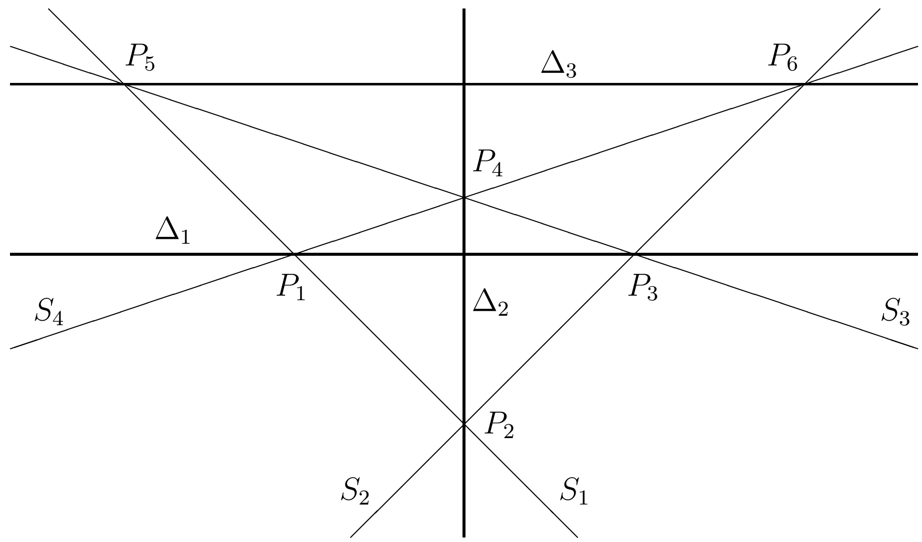

Figure 1: The quadrilateral $P_1P_2P_3P_4$ in $\mathbb{P}^2$

we have introduced satisfy the following relations:

(i) $-K_\Sigma \equiv \Delta_1 + \Delta_2 + \Delta_3$;

(ii) $f_i \equiv \Delta_{i+1} + \Delta_{i+2}$ for all $i \in \mathbb{Z}_3$;

(iii) $\Delta_i S_j = 0$ for all i, j;

(iv) $\Delta_i f_j = 2\delta_{ij}$ for $1 \leq i, j \leq 3$.

Denote by $\gamma_1, \gamma_2, \gamma_3$ the nonzero elements of $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and by $\chi_i \in \Gamma^*$ the nontrivial character orthogonal to γ_i; by [12, Propositions 2.1 and 3.1], to define a smooth Γ-cover $\pi \colon X \to \Sigma$, we specify:

(I) smooth divisors D_i for $i = 1, 2, 3$ such that $D = D_1 + D_2 + D_3$ is a normal crossing divisor,

(II) line bundles L_1, L_2 satisfying $2L_1 \equiv D_2 + D_3$, $2L_2 \equiv D_1 + D_3$.

The branch locus of π is D. More precisely, D_i is the image of the divisorial part of the fixed locus of γ_i on S. We have

$$\pi_* \mathcal{O}_S = \mathcal{O}_\Sigma \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1},$$

where $L_3 = L_1 + L_2 - D_3$, and Γ acts on L_i^{-1} via the character χ_i.

Here we set:

(I) $D_1 = \Delta_1 + f_2 + S_1 + S_2$, $D_2 = \Delta_2 + f_3$, $D_3 = \Delta_3 + f_1 + f_1' + S_3 + S_4$; where $f_1, f_1' \in |f_1|$, $f_2 \in |f_2|$, $f_3 \in |f_3|$ are general curves;

(II) $L_1 = 5l - e_1 - 2e_2 - e_3 - 3e_4 - 2e_5 - 2e_6$, and

$L_2 = 6l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6$

and we obtain $L_3 = 4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6$. For $i = 1, \dots, 4$, the (set theoretic) inverse image of S_i in X is the disjoint union of two -1-curves E_{i1}, E_{i2}; contracting these 8 exceptional curves on X and contracting the S_i on Σ, we obtain a smooth $\mathbb{Z}_2^2$-cover $p\colon S \to V$. The map p is branched on the four singular points of V and on the image $\overline{D}$ of D, which is contained in the smooth locus of V. The bicanonical divisor $2K_X$ is equal to $\pi^*(2K_\Sigma + D) = \pi^*(-K_\Sigma + f_1 + S_1 + S_2 + S_3 + S_4) = \pi^*(-K_\Sigma + f_1) + 2\sum E_{ij}$, and thus the bicanonical divisor $2K_S$ is equal to $\pi^*(-K_V + \overline{f}_1)$, where $\overline{f}_1$ is the image of f_1 in V. So $2K_S$ is ample, since it is the pullback of an ample line bundle by a finite map, S is minimal and of general type, and $K_S^2 = \frac{1}{4} 4(K_V + \overline{f}_1)^2 = 7$.

To compute the geometric genus of S, recall that $p_g(X) = p_g(\Sigma) + \sum h^0(\Sigma, K_\Sigma + L_i)$ (cf. [4] or [12, Lemma 4.2]). We have

$$\begin{aligned} K_\Sigma + L_1 &= 2l - e_2 - 2e_4 - e_5 - e_6, \\ K_\Sigma + L_2 &= 3l - e_1 - e_2 - e_3 - e_4 - 2e_5 - 2e_6, \\ K_\Sigma + L_3 &= l - e_1 - e_2 - e_3. \end{aligned}$$

We show that $h^0(\Sigma, K_\Sigma + L_2) = 0$. Assume by contradiction that there exists $D \in |K_\Sigma + L_2|$ and consider the image C of D in $\mathbb{P}^2$; C is a cubic containing $P_1, \dots, P_6$ which has a double point at P_5 and P_6. By Bezout's theorem, Δ_3 is contained in C and thus $C = \Delta_3 + Q$, where Q is a conic containing $P_1, \dots, P_6$, which is impossible. By similar (easier) arguments, one shows that $h^0(\Sigma, K_\Sigma + L_1) = h^0(\Sigma, K_\Sigma + L_3) = 0$, and thus $p_g(S) = p_g(X) = 0$. By the projection formula for a finite flat morphism,

$$H^0(X, 2K_X) = H^0(\Sigma, -K_\Sigma + f_1 + \sum S_j) \oplus \Big(\bigoplus_i H^0(\Sigma, -K_\Sigma + f_1 + \sum S_j - L_i)\Big),$$

and Γ acts on $H^0(\Sigma, -K_\Sigma + f_1 + \sum S_j - L_i)$ via the character χ_i. We have $h^0(\Sigma, -K_\Sigma + f_1 + \sum S_j) = h^0(\Sigma, -K_\Sigma + f_1)$, since

$$S_j(-K_\Sigma + f_1 + \sum S_i) = -2 \quad \text{for } i = 1, \dots, 4;$$

in addition, $h^0(\Sigma, -K_\Sigma + f_1) = 7$, since Σ is rational, $2f_1 + f_2 + f_3$ has arithmetic genus 7, and $-K_\Sigma + f_1 = K_\Sigma + 2f_1 + f_2 + f_3$. Since $p_2(S) = 8$, there is a value $i \in \{1, 2, 3\}$ such that $h^0(\Sigma, -K_\Sigma + \sum S_j + f_1 - L_i) = 1$ and $h^0(\Sigma, -K_\Sigma + \sum S_j + f_1 - L_k) = 0$ for $k \neq i$. Actually, an argument similar to that used for computing $p_g(S)$ shows that

$$\begin{aligned}
h^0(-K_\Sigma + \sum S_j + f_1 - L_1) &= h^0(\sum S_j + e_4) = 1,\\
h^0(-K_\Sigma + \sum S_j + f_1 - L_2) &= h^0(3l - e_1 - 2e_2 - e_3 - 2e_4 - e_5 - e_6) = 0,\\
h^0(-K_\Sigma + \sum S_j + f_1 - L_3) &= h^0(5l - e_1 - 2e_2 - e_3 - 3e_4 - 3e_5 - 3e_6) = 0.
\end{aligned}$$

It follows that the bicanonical map $\varphi\colon S \to \mathbb{P}^7$ is composed with the involution γ_1 but not with γ_2 and γ_3. Since $|2K_S| \supset \pi^*|-K_\Sigma|$ and the map $\Sigma \to \mathbb{P}^3$ induced by $|-K_\Sigma|$ is birational, it follows that φ has degree 2.

The remaining examples are obtained using the following construction due to Beauville (see [1, p. 123, Ex. 4] and cf. [7]). Let C_1, C_2 be curves of genus g_1, g_2, and assume that a group G of order $(g_1 - 1)(g_2 - 1)$ acts on C_1, C_2 so that C_i/G is isomorphic to $\mathbb{P}^1$ for $i = 1, 2$; write $p_i\colon C_i \to \mathbb{P}^1$ for the projections onto the quotients and $p\colon C_1 \times C_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ for the product of p_1 and p_2. Thus p is a Galois cover with group $G \times G$. Assume in addition that there exists an automorphism $\psi \in \operatorname{Aut} G$ whose graph $\Gamma = \Gamma_\psi \subset G \times G$ acts freely on $C_1 \times C_2$. Then set $S = (C_1 \times C_2)/\Gamma$ and denote by $q\colon C_1 \times C_2 \to S$ the quotient map and by $\pi\colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ the map induced by p. If G is Abelian, then π is a G-cover. The surface S is minimal and of general type since $C_1 \times C_2$ is minimal of general type and q is étale. Since Γ acts freely, $\chi(\mathcal{O}_{C_1 \times C_2}) = |G|\chi(\mathcal{O}_S)$ and $K^2_{C_1 \times C_2} = |G| K^2_S$, namely $\chi(\mathcal{O}_S) = 1$, $K^2_S = 8$. The irregularity $q(S)$ equals the dimension of the Γ-invariant subspace of

$H^0(C_1\times C_2,\Omega^1_{C_1\times C_2})\cong H^0(C_1,\omega_{C_1})\oplus H^0(C_2,\omega_{C_2})$. Since C_1/G and C_2/G are both rational and ψ is an automorphism, it follows that $q(S)=0$, and thus $p_g(S)=0$.

Example 4.2 As far as we know, this is a new example. In this case $G=\mathbb{Z}_2^3$, $g_1=5$, $g_2=3$. We denote by $\gamma_1,\gamma_2,\gamma_3$ the standard generators of G and by χ_1,χ_2,χ_3 the dual basis of the group of characters G^*. To construct the G-cover $p_i\colon C_i\to\mathbb{P}^1$ we have to specify (cf. [12, Propositions 2.1 and 3.1]):

(i) a divisor D_γ on $\mathbb{P}^1$ for each nonzero $\gamma\in G$;

(ii) line bundles L_1,L_2,L_3 on $\mathbb{P}^1$ satisfying

$$2L_i\equiv\sum_\gamma \varepsilon_i(\gamma)D_\gamma,\quad\text{where}\quad\begin{cases}\varepsilon_i(\gamma)=0 & \text{if } \gamma\in\ker\chi_i,\\ \varepsilon_i(\gamma)=1 & \text{otherwise.}\end{cases}$$

To construct $p_1\colon C_1\to\mathbb{P}^1$, we choose distinct points $P_1,\dots,P_6\in\mathbb{P}^1$ and set $D_{\gamma_1}=P_1+P_2$, $D_{\gamma_2}=P_3+P_4$, $D_{\gamma_3}=P_5+P_6$, $D_\gamma=0$ for $\gamma\neq\gamma_i$, and $L_1=L_2=L_3=\mathcal{O}_{\mathbb{P}^1}(1)$. The curve C_1 is smooth connected of genus 5. To construct $p_2\colon C_2\to\mathbb{P}^1$, we choose distinct points $Q_1,\dots,Q_5\in\mathbb{P}^1$ and set $D_{\gamma_1}=Q_1$, $D_{\gamma_2}=Q_2$, $D_{\gamma_1+\gamma_2}=Q_3$, $D_{\gamma_3}=Q_4+Q_5$, $D_\gamma=0$ for the remaining nonzero elements of G, and $L_1=L_2=L_3=\mathcal{O}_{\mathbb{P}^1}(1)$. The curve C_2 is smooth connected of genus 3. Define $\psi\in\operatorname{Aut}G$ by

$$\gamma_1\mapsto\gamma_1+\gamma_3,\quad\gamma_2\mapsto\gamma_2+\gamma_3,\quad\gamma_3\mapsto\gamma_1+\gamma_2+\gamma_3.$$

In the above notation, $\pi\colon S\to\mathbb{P}^1\times\mathbb{P}^1$ is a G-cover and $2K_S=\pi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1)$. By the projection formula, we have

$$H^0(S,2K_S)=\bigoplus_{\chi\in\Gamma^\perp}H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1)\otimes M_\chi^{-1}),$$

where M_χ^{-1} is the eigensheaf of $\pi_*\mathcal{O}_S$ corresponding to $\chi\in\Gamma^\perp\cong G^*$, and $(G\times G)/\Gamma\cong G$ acts on $H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1)\otimes M_\chi^{-1})$ via χ. The M_χ are line bundles on $\mathbb{P}^1\times\mathbb{P}^1$ that can be determined using [12, (2.15)]. One checks that $H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1)\otimes M_\chi^{-1})$ is nonzero only for the elements of $\Gamma^\perp$ that are orthogonal to $(0,\gamma_3)\in G\times G$. It follows that the bicanonical map of S is composed with the involution induced on S by $(0,\gamma_3)$, and thus it has degree 2 by Theorem 1.1.

Example 4.3 As for Example 4.1, this is due to Inoue [8, p. 317], and arises as the quotient of a complete intersection in the product of 4 elliptic curves by a free group action. Here we give a construction in the style of Beauville as explained above which is more suitable for our purpose. Let $\gamma_1, \ldots, \gamma_4$ be a basis of $G = \mathbb{Z}_2^4$ and $\chi_1, \ldots, \chi_4$ the dual basis of G^*; set $\gamma_0 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. We construct C_i as G-covers of $\mathbb{P}^1$ for $i = 1, 2$. As in Example 4.2, for this, we specify (cf. [12, Propositions 2.1 and 3.1]):

(i) a divisor D_γ of $\mathbb{P}^1$ for every nonzero $\gamma \in G$;

(ii) line bundles $L_1, \ldots, L_4$ of $\mathbb{P}^1$ satisfying

$$2L_i \equiv \sum_\gamma \varepsilon_i(\gamma) D_\gamma, \quad \text{where} \quad \begin{cases} \varepsilon_i(\gamma) = 0 & \text{if } \gamma \in \ker \chi_i \\ \varepsilon_i(\gamma) = 1 & \text{otherwise.} \end{cases}$$

Choose distinct points $P_0, \ldots, P_4 \in \mathbb{P}^1$ and set $D_{\gamma_i} = P_i$ for $i = 0, \ldots, 4$, $D_\gamma = 0$ if $\gamma \neq \gamma_i$, and $L_i = \mathcal{O}_{\mathbb{P}^1}(1)$ for $i = 1, \ldots, 4$. We write $p_1 \colon C_1 \to \mathbb{P}^1$ for the corresponding G-cover. Then C_1 is a smooth connected curve of genus 5. We construct the curve C_2 in the same way, starting from points $Q_0, \ldots, Q_4 \in \mathbb{P}^1$.

Let $\psi \in \operatorname{Aut} G$ be the automorphism:

$$\gamma_1 \mapsto \gamma_1 + \gamma_3, \quad \gamma_2 \mapsto \gamma_2 + \gamma_4, \quad \gamma_3 \mapsto \gamma_1 + \gamma_4, \quad \gamma_4 \mapsto \gamma_1 + \gamma_3 + \gamma_4.$$

In the above notation, $\pi \colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ is a G-cover and $2K_S = \pi^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$. Since $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ is very ample, it follows that the bicanonical map φ of S is birational if and only if it is not composed with an involution γ of the Galois group G of π. To check that this is indeed the case, we use the projection formula

$$H^0(S, 2K_S) = \bigoplus_{\chi \in \Gamma^\perp} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \otimes M_\chi^{-1}),$$

where M_χ^{-1} is the eigensheaf of $\pi_* \mathcal{O}_S$ corresponding to $\chi \in \Gamma^\perp \cong G^*$, and $(G \times G)/\Gamma \cong G$ acts on $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \otimes M_\chi^{-1})$ via χ. The M_χ are line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ that can be determined using [12, (2.15)]. It turns out that no $\gamma \in G \setminus \{0\}$ acts trivially on $H^0(S, 2K_S)$ and thus φ is birational.

Example 4.4 This example is due to Beauville and appears in [1, p. 123, Ex. 4], where the group action is not described explicitly, and in [7]. The assertion concerning the group action in [7] is not correct, since the group action described is not free.

In this case $g = 6$, $G = \mathbb{Z}_5^2$, and $C_1 = C_2 = \{x^5 + y^5 + z^5 = 0\} \subset \mathbb{P}^2$ is the Fermat quintic. If ε is a primitive 5th root of 1, then $(1,0) \in G$ acts on C by $(x : y : z) \mapsto (\varepsilon x : y : z)$ and $(0,1)$ acts by $(x : y : z) \mapsto (x : \varepsilon y : z)$. Let ψ be the automorphism of G taking $(1,0) \mapsto (1,-1)$ and $(0,1) \mapsto (1,2)$.

We compute the degree of the bicanonical map of S by writing down an explicit basis of the Γ-invariant subspace of $H^0(C \times C, 2K_{C\times C})$. Take homogeneous coordinates $(x : y : z;\ x_1 : y_1 : z_1)$ on $\mathbb{P}^2 \times \mathbb{P}^2 \supset C \times C$; using the fact that a regular 1-form on C is the residue of a rational form $\frac{g(x,y,z)}{x^5+y^5+z^5}\, dx \wedge dy \wedge dz$ for g homogeneous of degree 2, we see that $(a, b) \in G$ acts on bicanonical forms on $C \times C$ by:

$$\begin{aligned} x^i y^j z^{4-i-j} x_1^\alpha y_1^\beta z_1^{4-\alpha-\beta} &\mapsto \varepsilon^l x^i y^j z^{4-i-j} x_1^\alpha y_1^\beta z_1^{4-\alpha-\beta}, \\ &\text{where} \quad l = a(2 + i + \alpha - \beta) + b(3 + j + \alpha + 2\beta) \end{aligned}$$

Thus the following is a basis of $H^0(Y, 2K_Y)^{\mathrm{inv}}$:

$$\begin{gathered} x^4 y_1 z_1^3, \quad y^3 z y_1^2 z_1^2, \quad xyz^2 y_1^3 z_1, \quad x^2 yz x_1 z_1^3, \quad z^4 x_1 y_1^3, \\ xz^3 x_1^2 z_1^2, \quad x^3 y x_1^2 y_1^2, \quad y^4 x_1^3 z_1, \quad xy^2 z x_1^3 y_1. \end{gathered}$$

The subfield of $\mathbb{C}(S)$ generated by ratios of these monomials is the function field $\mathbb{C}(\Sigma)$ of the bicanonical image Σ of S. The map $\pi \colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ identifies $\mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ with the subfield of $\mathbb{C}(S)$ generated by $x^5 z^{-5}$ and $x_1^5 z_1^{-5}$. The extension $\mathbb{C}(S) \supset \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ is Galois with Galois group $G = \mathbb{Z}_5^2$. We observe:

$$x^5 z^{-5} = (x^3 y x_1^2 y_1^2)(x^4 y_1 z_1^3)(x^2 yz x_1 z_1^3)^{-1}(z^4 x_1 y_1^3)^{-1}$$

and

$$x_1^5 z_1^{-5} = (z^4 x_1 y_1^3)^2 (y^3 z y_1^2 z_1^2)(x^3 y x_1^2 y_1^2)^2 (x^2 yz x_1 z_1^3)^{-1}(xyz^2 y_1^3 z_1)^{-4}.$$

It follows that $\mathbb{C}(\Sigma) \supset \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$. Now one checks that no element of the Galois group $G = \mathbb{Z}_5^2$ of $\mathbb{C}(S)$ over $\mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ acts trivially on $\mathbb{C}(\Sigma)$. It follows that $\mathbb{C}(\Sigma) = \mathbb{C}(S)$, namely φ is birational.

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